Lecture 18:

Poisson Processes

1° **Definition 18.1.** We say that X has a Poisson distribution with mean λ, or $X = \text{Poisson}(\lambda)$, if

$$\mathbb{P}(X=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}, \quad \text{for } n = 0, 1, 2, \cdots$$

Recall that a binomial(m, p) distribution Y has density

$$\mathbb{P}(Y=k) = \binom{m}{k} p^k (1-p)^{m-k}.$$

Properties of Poission (λ).

Remark 18.1. $\mathbb{E}X = \lambda$, $\text{var}(X) = \lambda$.

Remark 18.2. The moment generating function of Possion distribution

$$\begin{aligned}
M[t] = \mathbb{E}[e^{tX}] &= \sum_{n=0}^{\infty} e^{tn} \cdot \mathbb{P}(X=n) \\
&= \sum_{n=0}^{\infty} e^{tn} \cdot e^{-\lambda} \cdot \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \cdot \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda^n}{n!} \\
&= e^{-\lambda} \cdot \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\
&= e^{-\lambda} \cdot e^{\lambda e^t} \\
&= e^{\lambda(e^t - 1)}.
\end{aligned}$$

Prepared by Kelvin Shuangjian ZHANG (ks3zhang@waterloo.ca)

Ex 18.1. For the Tim Hortons in DC, suppose the rate of arrivals is 60λ per hour. Suppose there are m students ($m \gg \lambda$) on campus today, the probability of each student go to this Tim Hortons at any one-minute time interval is $\frac{\lambda}{m}$. Thus, the probability that exactly k students will go during this minute is

$$\binom{m}{k} \cdot \left(\frac{\lambda}{m}\right)^k \cdot \left(1 - \frac{\lambda}{m}\right)^{m-k}.$$

Theorem 18.1. (limit of binomial is Poisson)

If m is large, then the binomial$(m, \frac{\lambda}{m})$ distribution is approximately Poisson(λ).

Proof. Let Y_m be the binomial$(m, \frac{\lambda}{m})$ and $X = \text{Poisson}(\lambda)$.

Then $\mathbb{P}(Y_m = k)$

$$= \frac{m!}{k!(m-k)!} \cdot \left(\frac{\lambda}{m}\right)^k \left(1 - \frac{\lambda}{m}\right)^{m-k}$$

$$= \frac{\lambda^k}{k!} \cdot \frac{m\cdot(m-1)\cdots(m-k+1)}{\underbrace{m\cdot m\cdot\cdots\cdot m}_{\text{"k" times}}} \cdot \left(1-\frac{\lambda}{m}\right)^m \cdot \left(1-\frac{\lambda}{m}\right)^{-k}.$$

Notice that ①. $\lim\limits_{m\to\infty} \frac{m\cdot(m-1)\cdots(m-k+1)}{\underbrace{m\cdot m\cdot\cdots\cdot m}_{\text{"k" times}}}$

$$= \lim_{m\to\infty}\frac{m}{m}\cdot\lim_{m\to\infty}\frac{m-1}{m}\cdot\cdots\cdot\lim_{m\to\infty}\frac{m-k+1}{m}$$

$$= \underbrace{1\cdot 1\cdot\cdots\cdot 1}_{\text{"k" times}}$$

$$= 1.$$

②. $\lim\limits_{m\to\infty}\left(1-\frac{\lambda}{m}\right)^m$

$$= \lim_{m\to\infty}\left[\left(1-\frac{\lambda}{m}\right)^{-\frac{m}{\lambda}}\right]^{(-\lambda)}$$

$$= \left\{\lim_{m\to\infty}\left[1+\left(-\frac{\lambda}{m}\right)\right]^{-\frac{m}{\lambda}}\right\}^{(-\lambda)}$$

$$= e^{-\lambda}.$$

③. $\lim\limits_{m\to\infty}\left(1-\frac{\lambda}{m}\right)^k = \left[\lim\limits_{m\to\infty}\left(1-\frac{\lambda}{m}\right)\right]^k = 1^k = 1.$

Thus,

$$\lim_{m\to\infty}\mathbb{P}(Y_m=k) = \frac{\lambda^k}{k!}\cdot e^{-\lambda} = \mathbb{P}(X=k). \quad \square$$

Ex 1 (cont.) The probability that j students will go between 9:31 to 9:32 and k students will go between 9:32 to 9:34 is

$$\mathbb{P}(Y_m = j,\ Z_m = k)$$

$$= \binom{m}{j}\cdot\left(\frac{\lambda}{m}\right)^j\left(1-\frac{\lambda}{m}\right)^{m-j}\cdot\binom{m}{k}\cdot\left(\frac{2\lambda}{m}\right)^k\left(1-\frac{2\lambda}{m}\right)^{m-k}$$

$$= \mathbb{P}(Y_m = j)\cdot\mathbb{P}(Z_m = k)$$

Thus, $\lim_{m\to\infty} \mathbb{P}(Y_m = j,\ Z_m = k)$

$$= \lim_{m\to\infty} \mathbb{P}(Y_m = j)\cdot\lim_{m\to\infty}(Z_m = k)$$

$$= \mathbb{P}(\text{Poisson}(\lambda) = j)\cdot\mathbb{P}(\text{Poisson}(2\lambda) = k).$$

Therefore, the number of arrivals in two time intervals we choose are independent Poisson with means λ and 2λ.

2°. Definition 18.2. (Poisson Process)

Let $N(t)$ represents the total number of occurance or events that have happened up to and including time t. We say that $\{N(s),\ s \geq 0\}$ is a (homogeneous) Poisson Process with rate λ if

(i). $N(0) = 0$,

(ii). $N(t+s) - N(s) = \text{Poisson}(\lambda t)$, and

(iii). $N(t)$ has independent increments,

i.e., if $t_0 < t_1 < t_2 < \cdots < t_n$,

then $N(t_1) - N(t_0),\ N(t_2) - N(t_1),\ \cdots,\ N(t_n) - N(t_{n-1})$ are mutually independent.

Theorem 18.2 (Sum of independent Poisson distributions is a Poisson distribution).

If X_i are independent $\text{Poisson}(\lambda_i)$, then

$$X_1 + X_2 + \cdots + X_k = \text{Poisson}(\lambda_1 + \lambda_2 + \cdots + \lambda_k).$$

Proof. It is sufficient to prove the case $k=2$, since

why? the general result follows by mathematical induction.

$$
\begin{aligned}
&\mathbb{P}(X_1+X_2=n)\\
&=\sum_{m=0}^{n}\mathbb{P}(X_1=m)\cdot\mathbb{P}(X_2=n-m)\\
&=\sum_{m=0}^{n}e^{-\lambda_1}\cdot\frac{\lambda_1^m}{m!}\cdot e^{-\lambda_2}\cdot\frac{\lambda_2^{n-m}}{(n-m)!}\\
&=\sum_{m=0}^{n}e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}\cdot\frac{n!}{m!\,(n-m)!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}\\
&=e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}\cdot\sum_{m=0}^{n}\binom{n}{m}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^m\cdot\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-m}\\
&=e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}\cdot\left(\frac{\lambda_1}{\lambda_1+\lambda_2}+\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^n\\
&=e^{-(\lambda_1+\lambda_2)}\cdot\frac{(\lambda_1+\lambda_2)^n}{n!}\\
&=\mathbb{P}(\text{Poisson}(\lambda_1+\lambda_2)=n)
\end{aligned}
$$

Thus, $X_1+X_2=\text{Poisson}(\lambda_1+\lambda_2)$. □

Lemma 18.1. If two random variables have the same moment generating functions, then they have the same probability distribution.

That is, mathematically, if X and Y has moment generating functions on $(-r,r)$ for some $r>0$ and suppose the two moment generating functions are the same on this interval. Then

$$\mathbb{P}(X \le t) = \mathbb{P}(Y \le t), \quad \forall t \in \mathbb{R}.$$

Remark 18.3. Second proof of Theorem 18.2

By Lemma 18.1, we only need to show that X_1+X_2 has the same moment generating function with $X = \text{Poisson}(\lambda_1+\lambda_2)$. That is,

$$\begin{aligned}
M_{X_1+X_2}[t] &= \mathbb{E}[e^{t(X_1+X_2)}] = \mathbb{E}[e^{tX_1} \cdot e^{tX_2}] \\
&= \mathbb{E}[e^{tX_1}] \cdot \mathbb{E}[e^{tX_2}] \\
&= e^{\lambda_1(e^t-1)} \cdot e^{\lambda_2(e^t-1)} \\
&= e^{(\lambda_1+\lambda_2)(e^t-1)} \\
&= \mathbb{E}[e^{tX}] \quad = M_X[t]. \quad \square
\end{aligned}$$

By Remark 18.2

This is the end of this lecture!